TDPT 2 $\quad H = H_0 + H_1(t)$

PHYSICS 441 $\quad H_1(t) = 0$

$$|\psi(t)\rangle = \sum_n a_m |m\rangle e^{-iE_m t/\hbar}$$

WHEN $H_1(t) \neq 0$

$$|\psi(t)\rangle = \sum_m a_m(t) |m\rangle e^{-iE_m t/\hbar}$$

PROBLEM: FIND $a_n(t)$'s

EXACT SOLUTION

$$i\hbar \frac{d a_m(t)}{dt} = \sum_m a_m(t) \langle m|H_1|m\rangle$$

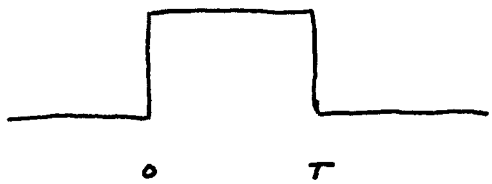
APPROXIMATION (FO) $\quad a_i(t=0) = 1$

all others $= 0$

$$a_m(T) = (i\hbar)^{-1} \int_0^T \langle m|H_1|i\rangle e^{-i\omega_{im} t} \, dt$$

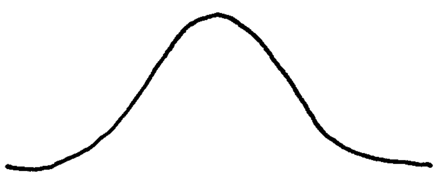
FOURIER TRANSFORMS

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

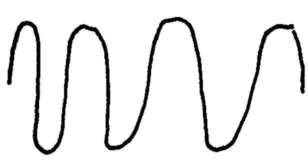


CONSTANT

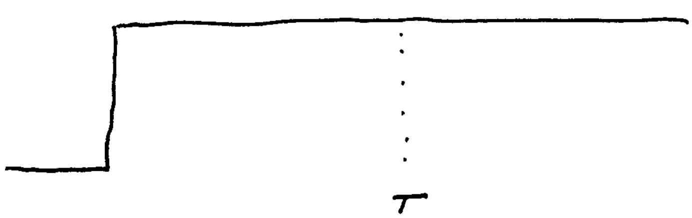
EXPONENTIAL



GAUSSIAN



HARMONIC



$$a_m(T) = (i\hbar)^{-1} \int_0^T \langle m | H_1 | i \rangle \, e^{-i\omega_{im} t} \, dt$$

$$= \frac{\langle m | H_1 | i \rangle}{i\hbar} \int_{-\infty}^{\infty} H_1(t) \, e^{-i\omega t} \, dt \Big|_{\omega = \omega_{im}}$$

↑ SPATIAL INTEGRAL

↑ TEMPORAL INTEGRAL

INTEGRATE OVER THE SPATIAL COMPONENT OF H_1

INTEGRATE OVER THE TEMPORAL COMPONENT OF H_1, THEN EVALUATE AT $\omega = \omega_{im}$

The Fourier Transform and Its Applications

Third Edition

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by substitution of variables. Hence the two theorems follow.

Derivative theorem The Fourier transform of $p'_\tau(x)$ is $i2\pi s P_\tau(s)$. Hence the Fourier transform of $p'(x)$ is $i2\pi s P(s)$.

Power theorem Since no meaning has been assigned to the product of two generalized functions, the best theorem that can be proved is

$$\int_{-\infty}^{\infty} P(s)\bar{F}(s)\,ds = \int_{-\infty}^{\infty} p(x)F(-x)\,dx,$$

where $F(x)$ is a particularly well-behaved function and $p(x)$ is a generalized function. The theorem follows from the fact that

$$\lim_{\tau\to 0}\int_{-\infty}^{\infty} P_\tau(s)\bar{F}(s)\,ds = \lim_{\tau\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{F}(s)p(x)e^{-i2\pi sx}\,dx\,ds$$
$$= \lim_{\tau\to 0}\int_{-\infty}^{\infty} p_\tau(x)F(-x)\,dx.$$

Summary of theorems

The theorems discussed in the preceding pages are collected for reference in Table 6.1.

Table 6.1 Theorems for the Fourier transform

Theorem	$f(x)$	$F(s)$
Similarity	$f(ax)$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$
Addition	$f(x) + g(x)$	$F(s) + G(s)$
Shift	$f(x - a)$	$e^{-i2\pi as}F(s)$
Modulation	$f(x)\cos\omega x$	$\frac{1}{2}F\left(s - \frac{\omega}{2\pi}\right) + \frac{1}{2}F\left(s + \frac{\omega}{2\pi}\right)$
Convolution	$f(x) * g(x)$	$F(s)G(s)$
Autocorrelation	$f(x) * f^*(-x)$	$\lvert F(s)\rvert^2$
Derivative	$f'(x)$	$i2\pi sF(s)$
Derivative of convolution	$\frac{d}{dx}[f(x) * g(x)] = f'(x) * g(x) = f(x) * g'(x)$	
Rayleigh	$\int_{-\infty}^{\infty} \lvert f(x)\rvert^2\,dx = \int_{-\infty}^{\infty} \lvert F(s)\rvert^2\,ds$	
Power	$\int_{-\infty}^{\infty} f(x)g^*(x)\,dx = \int_{-\infty}^{\infty} F(s)G^*(s)\,ds$	
(f and g real)	$\int_{-\infty}^{\infty} f(x)g(-x)\,dx = \int_{-\infty}^{\infty} F(s)G(s)\,ds$	

The dissociation into odd and even parts changes with changing origin of x, some functions such as $\cos x$ being convertible from fully even to fully odd by a shift of origin.

Significance of oddness and evenness

Let

$$f(x) = E(x) + O(x),$$

where E and O are in general complex. Then the Fourier transform of $f(x)$ reduces to

$$2 \int_0^\infty E(x) \cos (2\pi xs)\, dx - 2i \int_0^\infty O(x) \sin (2\pi xs)\, dx.$$

It follows that if a function is even, its transform is even, and if it is odd, its transform is odd. Full results are

Real and even	Real and even
Real and odd	Imaginary and odd
Imaginary and even	Imaginary and even
Complex and even	Complex and even
Complex and odd	Complex and odd
Real and asymmetrical	Complex and asymmetrical
Imaginary and asymmetrical	Complex and asymmetrical
Real even plus imaginary odd	Real
Real odd plus imaginary even	Imaginary
Even	Even
Odd	Odd

These properties are summarized in the following diagram:

$$f(x) = o(x) + e(x) = \text{Re } o(x) + i \text{ Im } o(x) + \text{Re } e(x) + i \text{ Im } e(x)$$

$$F(s) = O(s) + E(s) = \text{Re } O(s) + i \text{ Im } O(s) + \text{Re } E(s) + i \text{ Im } E(s).$$

Figure 2.5, which records the phenomena in another way, is also valuable for revealing at a glance the "relative sense of oddness": when $f(x)$ is real and odd with a *positive* moment, the odd part of $F(s)$ has i times a *negative* moment; and when $f(x)$ is real but not necessarily odd, we also find opposite senses of oddness. However, inverting the procedure—that is, going from $F(s)$ to $f(x)$, or taking $f(x)$ to be imaginary—produces the same sense of oddness.

Real even functions play a special part in this work because both they and their transforms may easily be graphed. Imaginary odd, real odd, and imaginary even functions are also important in this respect.

Another special kind of symmetry is possessed by a function $f(x)$ whose

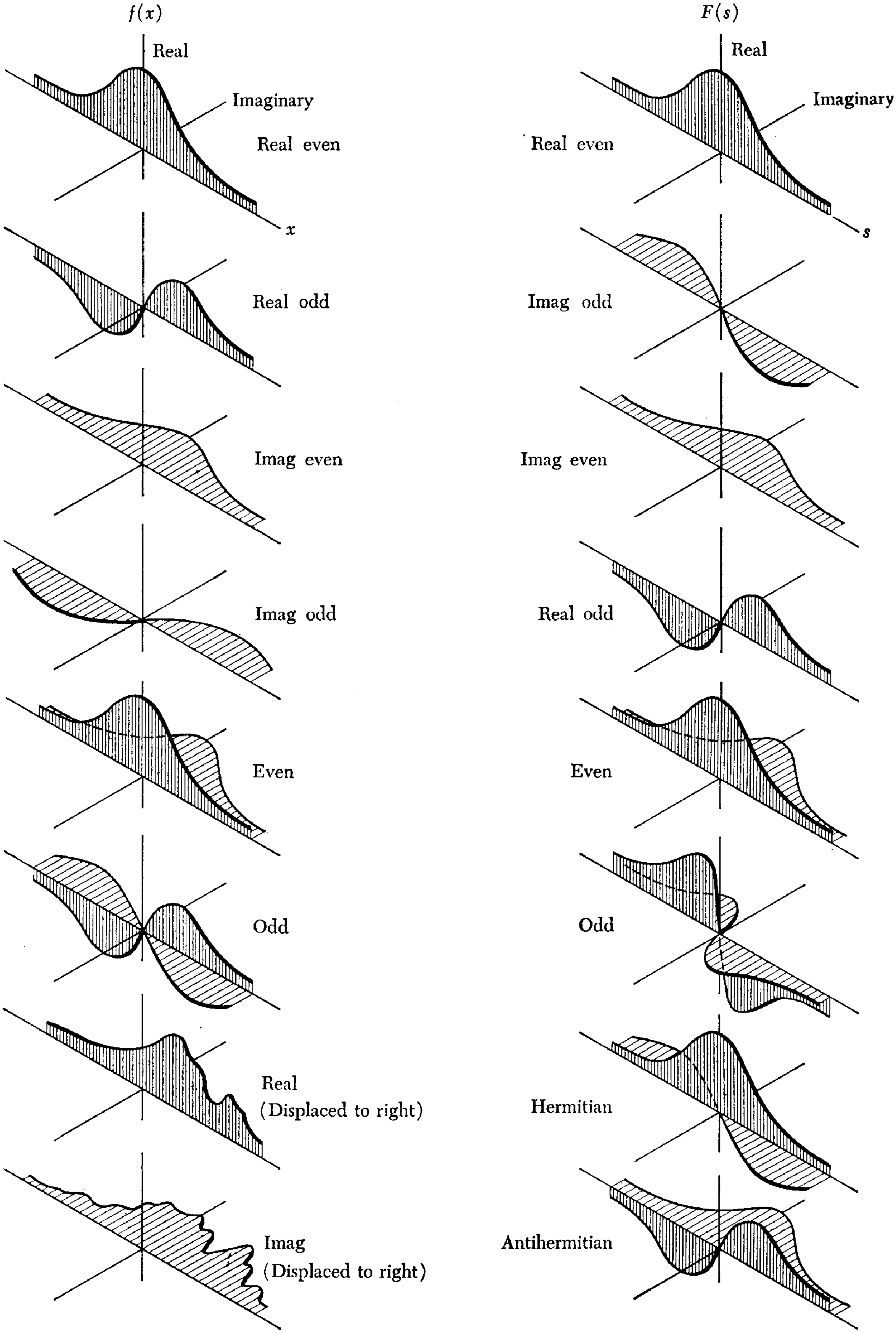
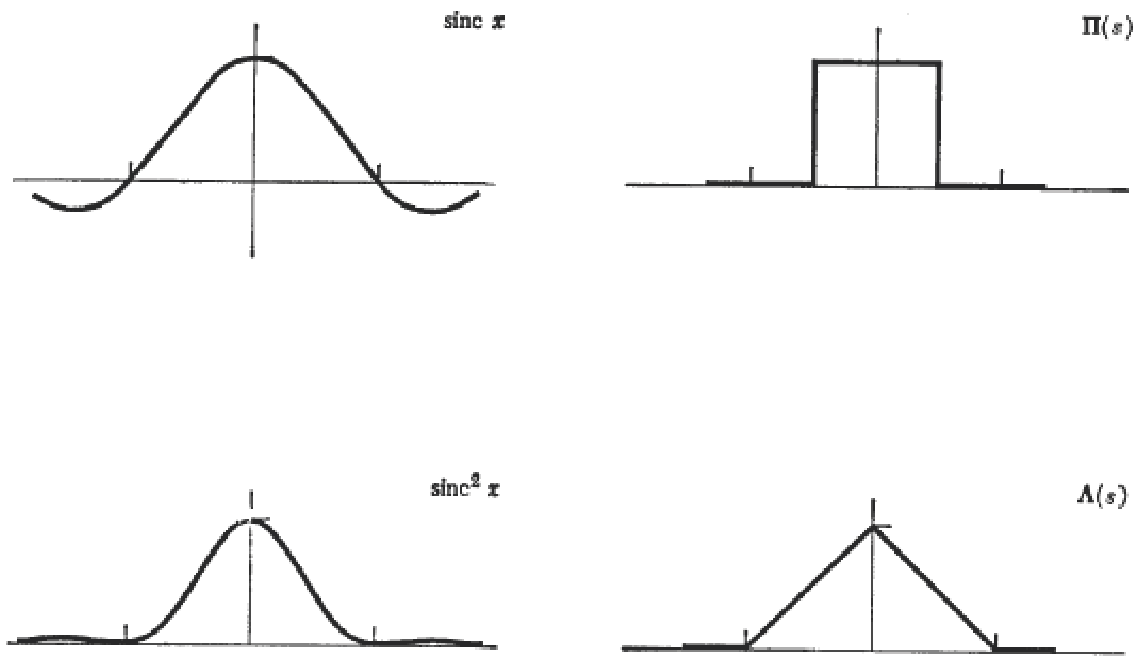
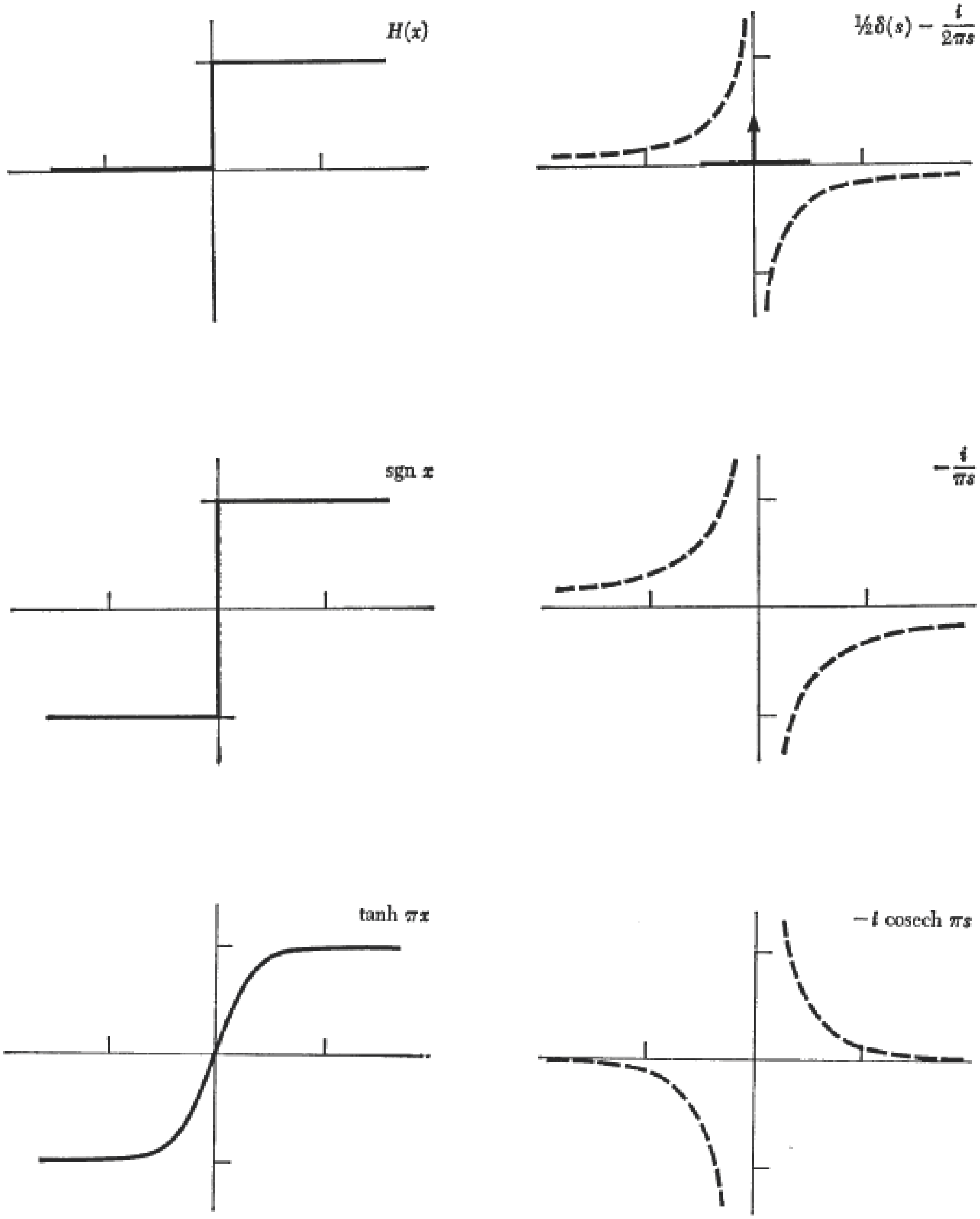


Fig. 2.5 Symmetry properties of a function and its Fourier transform.

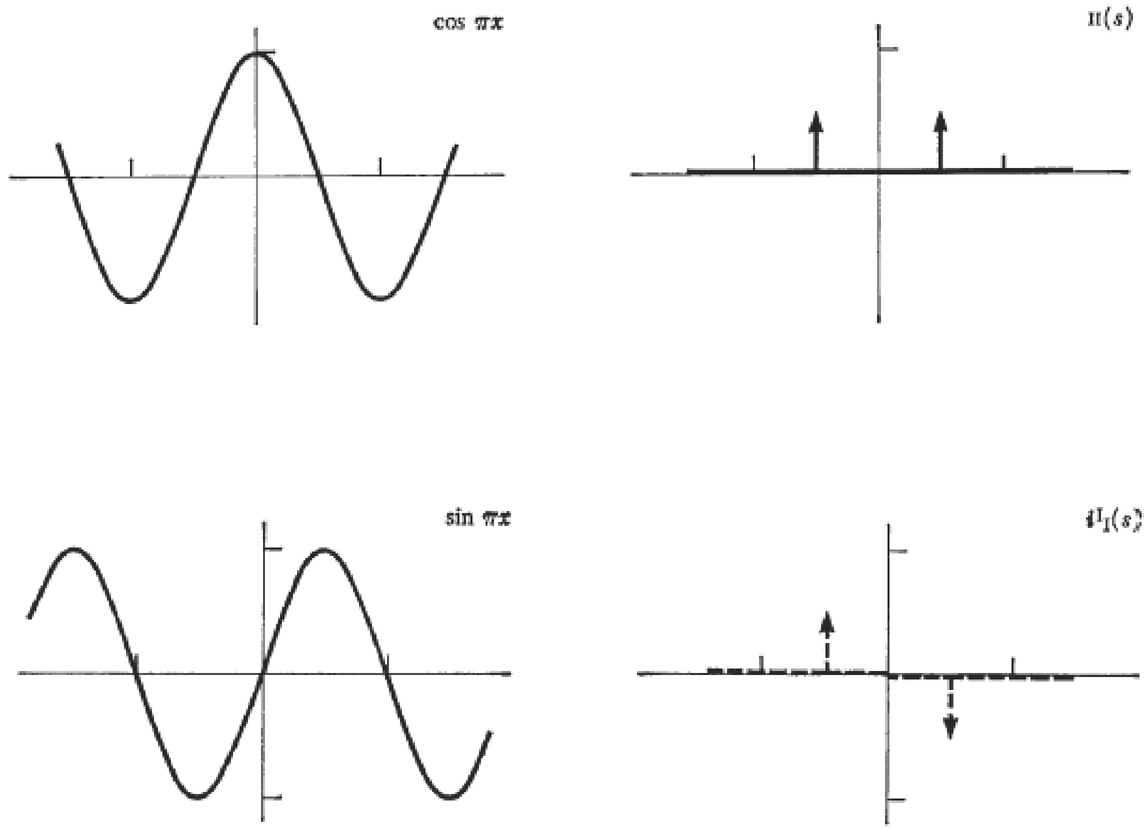
Sinc functions



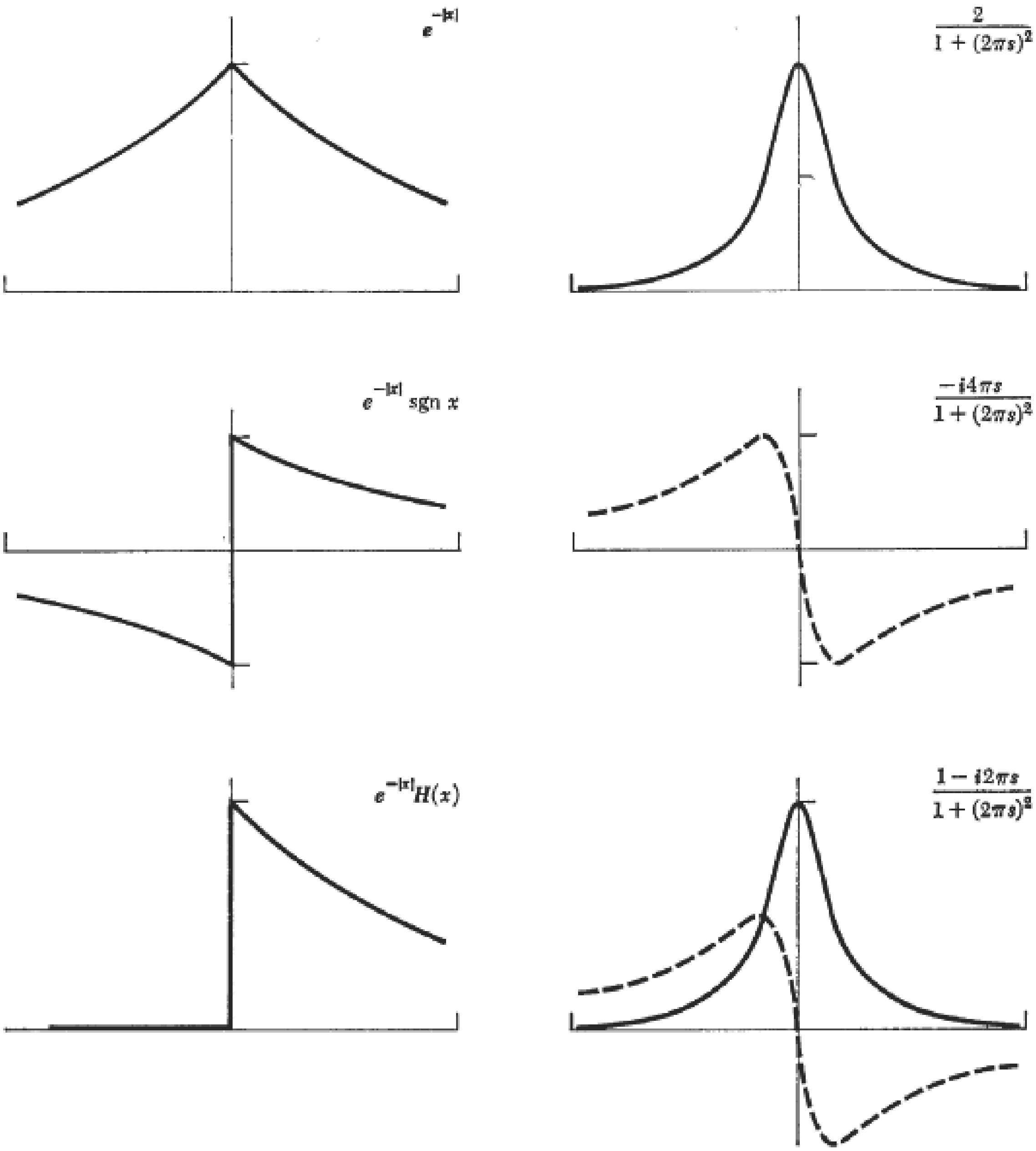
Step functions



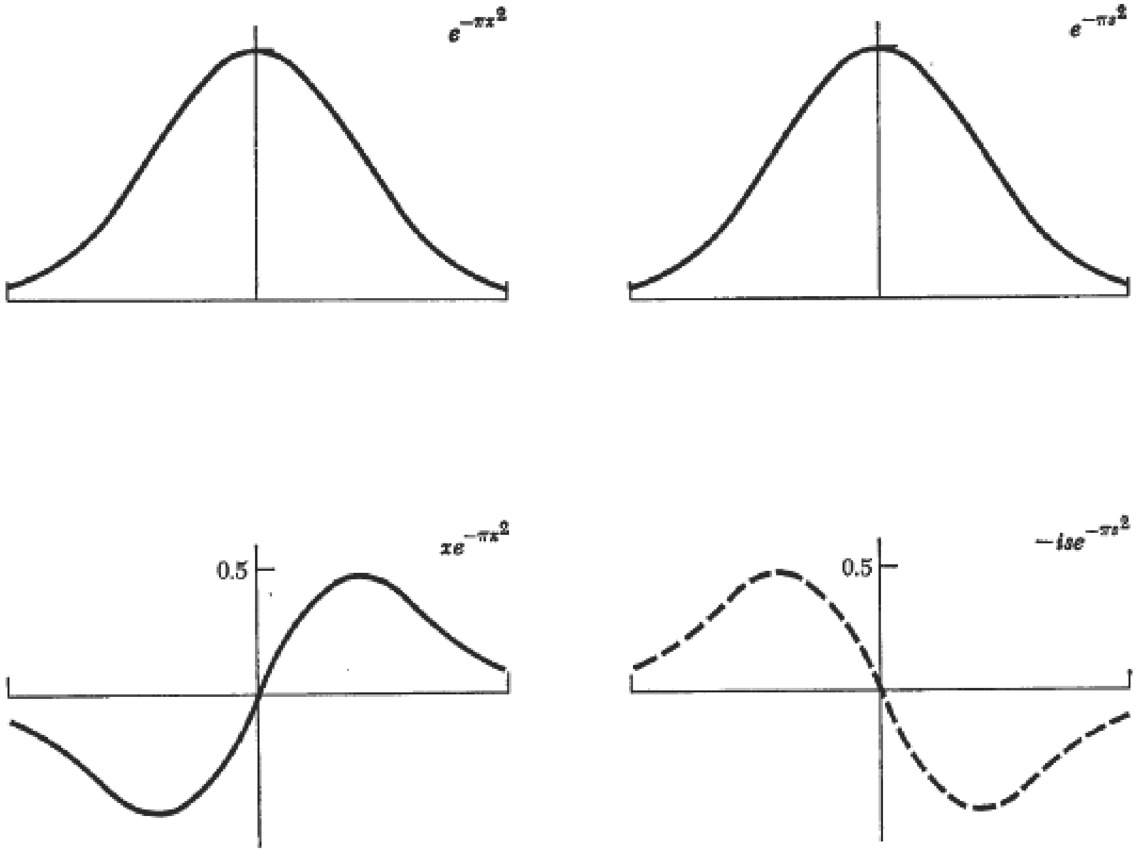
Sinusoidal functions



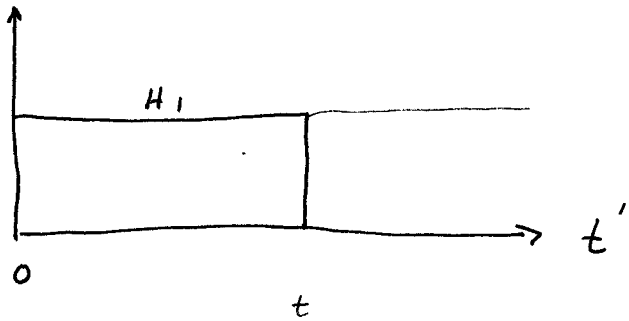
Exponential functions



Gaussian functions



(1) CONSTANT $H_1(t)$



$$a_m(t) = (i\hbar)^{-1} \int_0^t \langle m | H_1 | i \rangle e^{i\omega_{mi} t'} \, dt'$$

$$= \frac{\langle m | H_1 | i \rangle}{i\hbar} \int_0^t e^{i\omega_{mi} t'} \, dt'$$

$$= \frac{\langle m | H_1 | i \rangle}{(i\hbar)(i\omega_{mi})} \, e^{i\omega_{mi} t'} \Big|_{t'=0}^{t'=t}$$

$$a_m(t) = - \frac{\langle m | H_1 | i \rangle}{\hbar \omega_{mi}} \left[e^{i\omega_{mi} T} - 1 \right]$$

PROB $= |a_m(T)|^2$

$$|a_m(T)|^2 = |\langle m | H_1 | i \rangle|^2 \frac{\sin^2(\frac{1}{2}\omega_{mi} T)}{\hbar^2 (\frac{1}{2}\omega_{mi})^2}$$

because of [10–21b] four times as high. At a later time $3t_1$, the curve is three times narrower and nine times higher than the same curve at $t = t_1$. The area under the curve—which measures the total excitation in levels other than the kth—is thus increasing in proportion to t. The excitation "piles up" in those

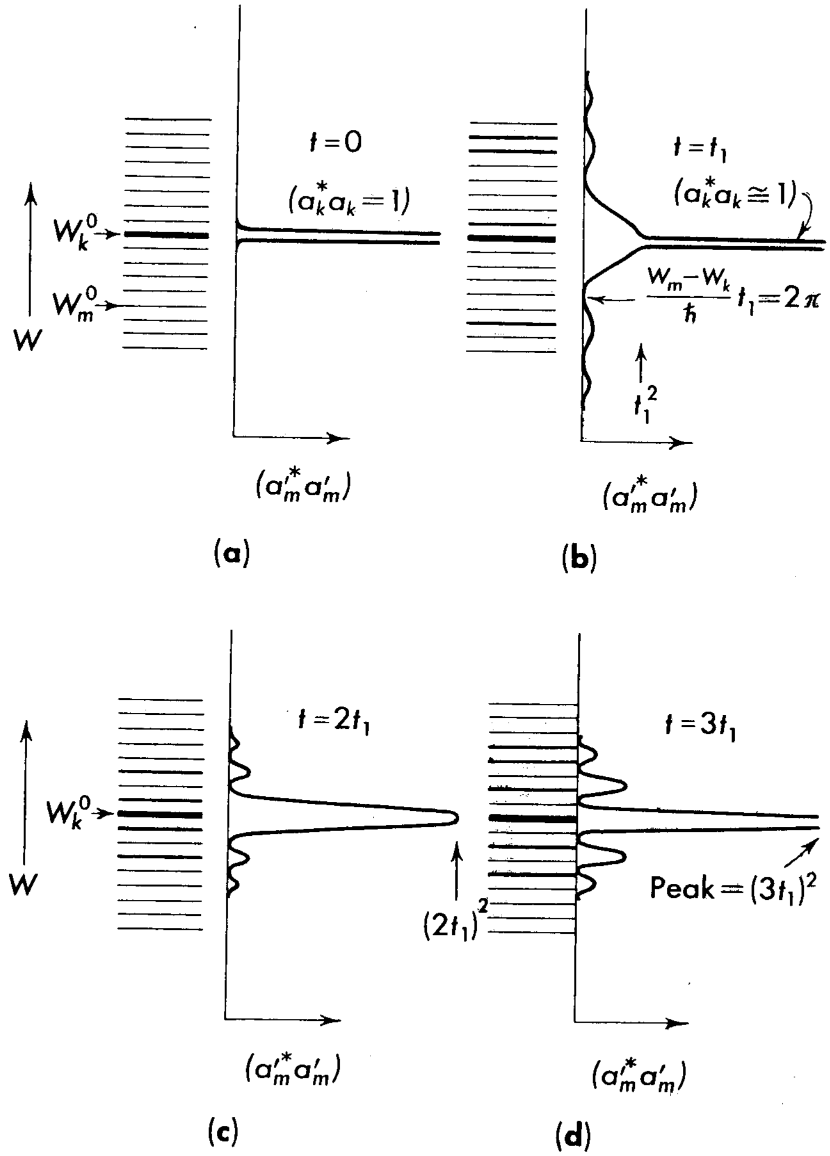


Fig. 10.2. The time variation of the excitation of the proper vibrations (eigenfunctions) caused by the constant perturbation, starting at $t = 0$. The density of the horizontal lines indicates the degree of excitation of the level or state.

levels nearest W_k^0, the effect being more pronounced the longer the perturbation is allowed to continue.

The detailed picture of the excitation process is complicated, except for those levels very near to W_k^0, which show a steady growth of excitation with t^2.

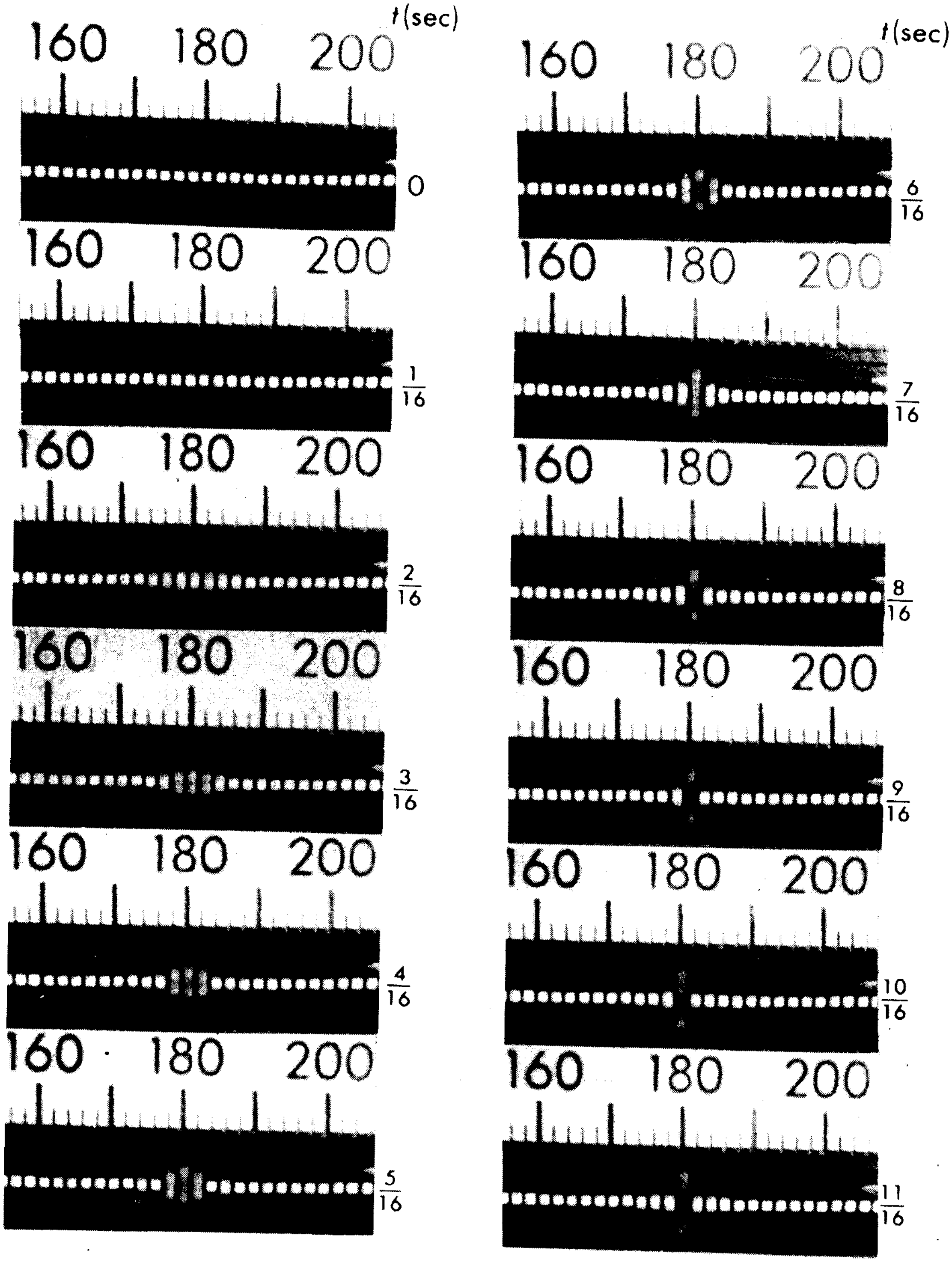


Fig. 10.4. Sequential photographs of a bank of reed filters. At $t = 0$, a constant-amplitude 180 cps signal is coupled equally to each of the reeds.

$$\left| e^{i\theta} - 1 \right|^2$$

$$|z|^2 = (z_R)^2 + (z_I)^2$$

$$\left| \cos\theta + i \sin\theta - 1 \right|^2 = (\cos\theta - 1)^2 + (\sin\theta)^2$$

$$= \cos^2\theta - 2\cos\theta + 1 + \sin^2\theta$$

$$= 2 - 2\cos\theta$$

$$= 2(1 - \cos\theta)$$

$$= 2\left(2\sin^2(\theta/2)\right)$$

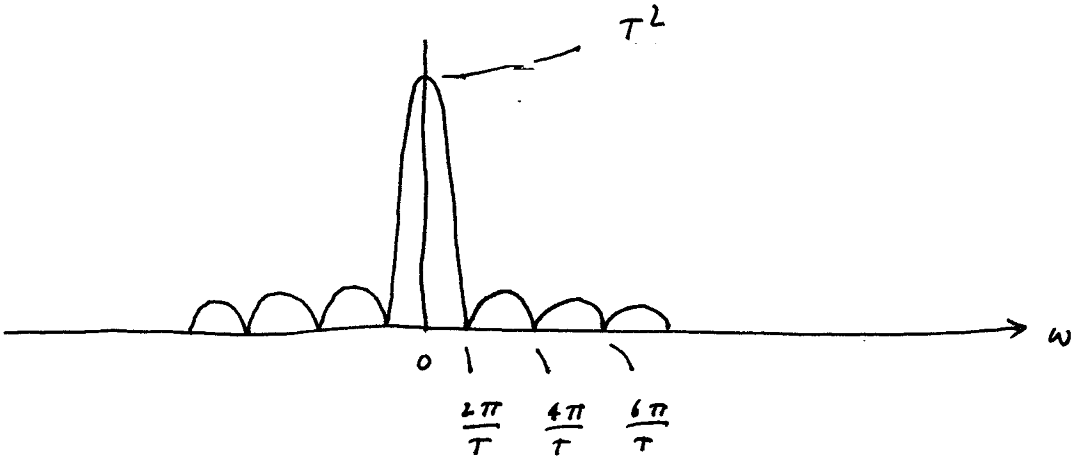
$$= 4\sin^2(\theta/2)$$

$\frac{\sin x}{x}$ called SINC

$$\frac{\sin^2\left(\frac{1}{2}\omega_{mi} T\right)}{\left(\frac{1}{2}\omega_{mi} T\right)^2}\, T^2$$

↖ $SINC^2$

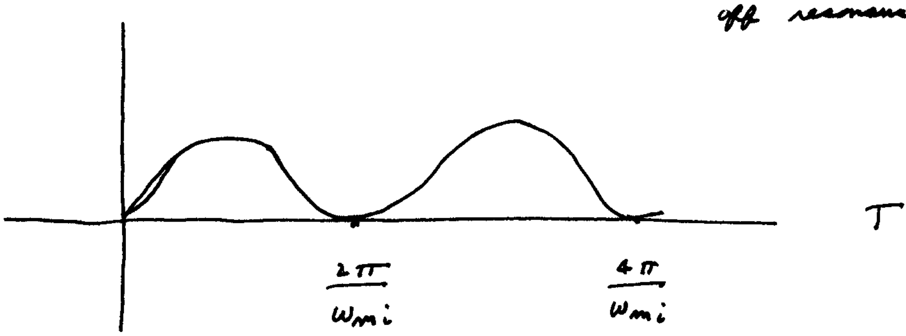
Frequency Dependence



$$\left.\begin{array}{ll} \text{HEIGHT} \sim & T^2 \\ \text{WIDTH} \sim & \frac{1}{T} \end{array}\right\} \Rightarrow \text{AREA} \sim T$$

TIME-DEPENDENCE

like a harmonic oscillator driven off resonance



ABOUT THE SINC FUNCTION

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$\text{SINC}(0) = 1$$

$$\text{SINC}(m) = 0$$

$$\int_{-\infty}^{\infty} \text{SINC}(x)\, dx = 1$$

as $T \to \infty$, $\text{SINC}(x) \to \delta(x)$

$$\text{SINC}^2(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2$$

$$\text{SINC}^2(0) = 1$$

$$\text{SINC}^2(m) = 0$$

$$\int_{-\infty}^{\infty} \text{SINC}^2(x)\, dx = 1$$

as $T \to \infty$, $\text{SINC}^2(x) \to \delta(x)\ \delta(x)$

(2) HARMONIC $\qquad H_1(t) = H_1 \cos(\omega t)$

$$a_m(T) = \left(\frac{1}{i\hbar}\right) \int_0^T \langle m | H_1 | i \rangle e^{i\omega_{mi} t} \cos(\omega t)\, dt$$

$\cos(\omega t) \rightarrow \frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t}$

$$= \frac{\langle m | H_1 | i \rangle}{2i\hbar} \int_0^T \left[e^{i(\omega_{mi}+\omega)t} + e^{i(\omega_{mi}-\omega)t} \right] dt$$

$$= \frac{\langle m | H_1 | i \rangle}{2i\hbar} \left[\frac{e^{i(\omega_{mi}+\omega)t} - 1}{\omega_{mi}+\omega} + \frac{e^{i(\omega_{mi}-\omega)t}}{\omega_{mi}-\omega} \right]_{t=0}^{t=T}$$

$$= -\frac{\langle m | H_1 | i \rangle}{2i\hbar} \left[\frac{e^{i(\omega_{mi}+\omega)T} - 1}{\omega_{mi}+\omega} + \frac{e^{i(\omega_{mi}-\omega)T} - 1}{\omega_{mi}-\omega} \right]$$

↑ ANTI RESONANT TERM $\qquad$ ↗ RESONANT TERM WINS!

If $E_m > E_i$ system absorbs energy $\qquad$ if $\omega \approx \omega_{mi}$

$$a_m(t) \approx -\frac{\langle m | H_1 | i \rangle}{2\hbar} \left[\frac{e^{i(\omega_{mi}-\omega)t} - 1}{\omega_{mi}-\omega} \right]$$

if $E_m < E_i$, system loses energy

$$\left| e^{i\theta} - 1 \right|^2 = 2(1 - \cos\theta) = 4\sin^2(\theta/2)$$

$$|a_m(T)|^2 = \frac{|\langle m | H_1 | i \rangle|^2}{\hbar^2} \frac{\sin^2[(\omega_{mi} - \omega)T/2]}{(\omega_{mi} - \omega)^2}$$

resonance regions are getting narrower (as $1/t$) and more intense at their maxima (as t^2). *Thus the total excitation of each resonance region grows in proportion to t, the duration of the perturbation.* (In these figures we assume, for con-

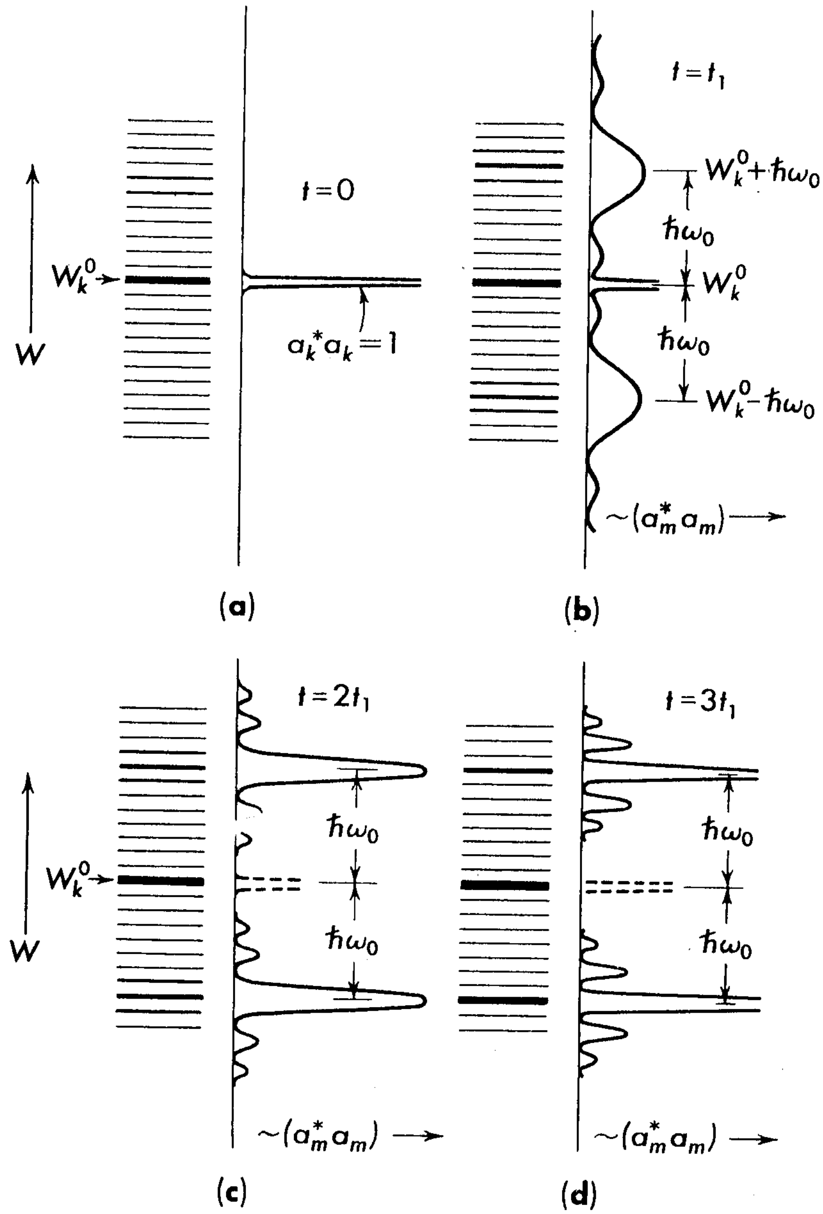
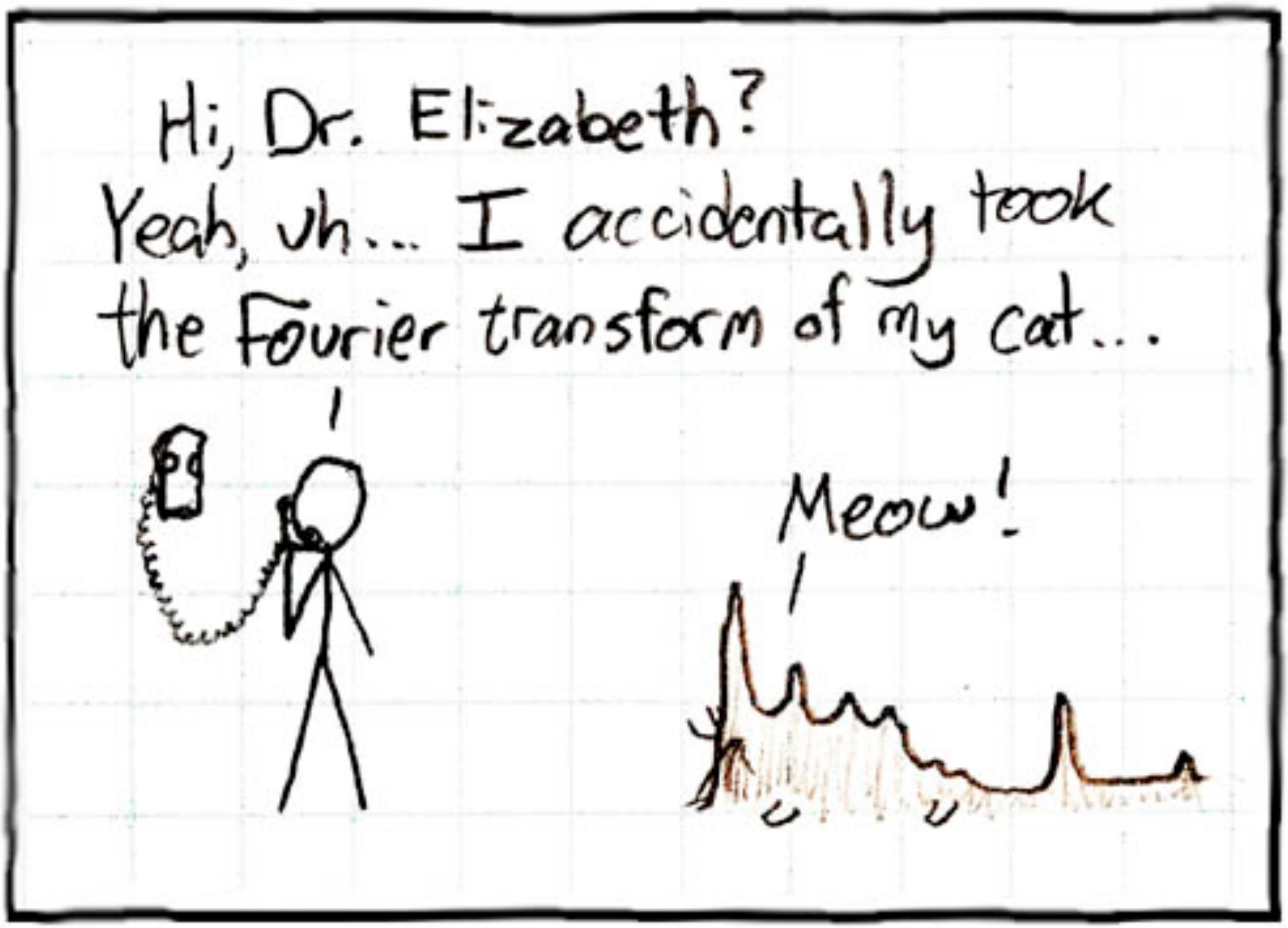


Fig. 10.3. The time variation of the excitation of the proper vibrations caused by a harmonic perturbation, starting at $t = 0$. The density of the horizontal lines indicates qualitatively the degree of excitation of the level or state.

venience, that the matrix elements connecting k to all other states are the same. Actually, of course, the matrix elements can, and do, exert a strong selective effect over and above the basic resonance effects. The matrix elements


Hi, Dr. Elizabeth?
Yeah, uh... I accidentally took the Fourier transform of my cat...
Meow!